Log-periodic corrections to scaling: exact results for aperiodic Ising quantum chains

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Abstract. Log-periodic amplitudes of the surface magnetization are calculated analytically for two Ising quantum chains with aperiodic modulations of the couplings. The oscillating behaviour is linked to the discrete scale invariance of the perturbations. For the Fredholm sequence, the aperiodic modulation is marginal and the amplitudes are obtained as functions of the deviation from the critical point. For the other sequence, the perturbation is relevant and the critical surface magnetization is studied.

1. Introduction

The possible occurrence of log-periodic critical amplitudes in systems with a discrete scale invariance has been known since the early days of the renormalization group in statistical mechanics [1–3].

Under length rescaling by a factor of b, in the case of a one-parameter renormalization group, the linearized recursion relation for the singular part of some observable $F(\theta)$ takes the form

$$F_{\text{sing}}(\theta) = \frac{1}{a} F_{\text{sing}}(\mu\theta) \tag{1.1}$$

where a and μ depend on the dilatation factor b. Looking for a power-law solution under the form $F_{\text{sing}}(\theta) = A(\theta)\theta^x$, one obtains $x = \ln a / \ln \mu$ and $A(\theta) = A(\mu\theta)$. The condition on the amplitude is satisfied by a periodic function of $\ln \theta$ with period $\ln \mu$. When the scale invariance is continuous, b remains arbitrary and cannot enter the amplitude which then reduces to a constant. In contrast, with a discrete scale invariance a dependence on the fixed scaling factor is allowed and the amplitude may be log-periodic. Such a behaviour may be ascribed to complex exponents since a Fourier component may be written as $\text{Re}\{\theta^{x+ix'}\} = \cos(x'\ln\theta)\theta^x$.

Oscillating amplitudes indeed appear in systems with built-in discrete scale invariance. One may mention low-dimensional dynamical models for the transition to chaos [4–8], wave propagation on discrete fractals [9], superconductive transitions on Sierpinsky networks [10] and spin systems on hierarchical lattices [11].

More recently, the same behaviour has also been observed in systems like fracture in heterogeneous media [12], foreshock activity preceding major earthquakes [13] and diffusion-limited aggregation [14], where the discrete scale invariance is not apparent. Such

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log-periodic modulations might be of great practical importance in refining the prediction of earthquakes [13, 15].

The purpose of the present work is to study analytically such oscillating amplitudes which appear in the surface critical behaviour of aperiodic Ising quantum chains with Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2} \sum_{k=1}^{\infty} [\sigma_k^z + \lambda_k \sigma_k^x \sigma_{k+1}^x] \qquad \lambda_k = \lambda r^{f_k} \tag{1.2}$$

where the σ_ks are Pauli spin operators, λ is the unperturbed coupling and r the modulation amplitude of the couplings. The aperiodic sequence, $f_k = 0$ or 1, is generated via an inflation rule, leading to a discrete scale invariance for the modulation.

Previous work on spin systems involved a linear approximation for the recursion relation (1.1), leading to errors of several decades in the oscillation amplitudes [11]. Here the recursion can be treated without any approximation and we obtain exact results for the log-periodic amplitude of the surface magnetization with two different modulations. We first complete the study of a marginal sequence for which the leading critical behaviour is known [16]. Then we consider a relevant aperiodic modulation, leading to a non-vanishing surface magnetization at the bulk critical point.

2. Marginal surface magnetization: the generalized Fredholm sequence

The generalized Fredholm sequence, which is the characteristic sequence of the powers of m, follows from substitutions on the letters A, B and C:

$$\begin{aligned} A &\to \mathcal{S}(A) = A\,B\,C\,C\,\ldots C \\ B &\to \mathcal{S}(B) = B\,C\,C\,C\,\ldots C \\ C &\to \mathcal{S}(C) = \underbrace{C\,C\,C\,C\,\ldots C}_{m}\,. \end{aligned} \tag{2.1}$$

With words of length $m = 2$, one recovers the usual Fredholm substitution [17]. Starting with A and associating $f_k = 0$ to A and C and $f_k = 1$ to B, for $m = 2$, one obtains the sequence

$$0\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\ldots. \tag{2.2}$$

More generally, $f_k = 1$ when $k = m^p + 1$ so that $n_j = \sum_{k=1}^{j} f_k$ satisfies the recursion relation $n_{ml+q} = n_{l+1} + 1$ ($q = 1, m$; $n_1 = 0$) which can be iterated to give $n_L \sim \ln L$. The density of perturbed couplings, n_L/L, vanishes asymptotically: the Fredholm aperiodicity is a surface-extended perturbation which does not affect the bulk critical properties of the system [16]. The critical coupling, in particular, remains at $\lambda_c = 1$.

A heuristic scaling argument has been developed recently by Luck [18], extending to aperiodic perturbations the Harris criterion for random systems [19]. According to Luck's criterion, the relevance of the perturbation is governed by the crossover exponent $\phi = 1 + \nu(\omega - 1)$ which involves the correlation length exponent ν (equal to 1 for the $d = 1 + 1$ Ising model) and the wandering exponent ω of the aperiodic sequence [20, 21]. For the Fredholm sequence, $\omega = 0$ so that $\phi = 0$ and the perturbation is marginal: the surface exponents vary continuously with the modulation amplitude r [16].

The surface magnetization m_s may be expressed as a series involving products of the couplings λ_k [22]

$$m_s = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \lambda_k^{-2}\right)^{-1/2} \tag{2.3}$$

which may be rewritten as

$$m_s = [S(\lambda, r)]^{-1/2} \qquad S(\lambda, r) = \sum_{j=0}^{\infty} \lambda^{-2j} r^{-2n_j} \qquad n_0 = 0. \tag{2.4}$$

With $T(\lambda, r) = S(\lambda, r) - 1 - \lambda^{-2}$, the following recursion is obtained [16]:

$$(\lambda^2 - 1)T(\lambda, r) = r^{-2}(\lambda^{-2} - \lambda^{-2m}) + r^{-2}(\lambda^{2m} - 1)T(\lambda^m, r). \tag{2.5}$$

In the recursion process, λ is changed into $R[\lambda] = \lambda^m$. Defining $\theta = \ln \lambda^2$, one obtains a linear renormalization in the new variable:

$$R[\theta] = m\theta. \tag{2.6}$$

One may notice that θ is a natural variable in the problem since, with $\lambda_c = 1$, it gives the deviation from the critical point:

$$\theta = \ln \left(\frac{\lambda}{\lambda_c} \right)^2 \simeq \left(\frac{\lambda}{\lambda_c} \right)^2 - 1. \tag{2.7}$$

With the new variable and

$$F(\theta) = (e^{\theta} - 1)T(e^{\theta/2}, r) \qquad \varphi(\theta) = r^{-2}(e^{-\theta} - e^{-m\theta}) \tag{2.8}$$

the recursion relation (2.5) reads

$$F(\theta) = \varphi(\theta) + r^{-2}F(m\theta) = \sum_{j=0}^{\infty} r^{-2j} \varphi(m^j \theta) \tag{2.9}$$

where the last expression follows from the iteration process.

Making use of the expansion

$$\varphi(\theta) = \sum_{k=1}^{\infty} \varphi_k \theta^k \qquad \varphi_k = \frac{(-1)^k}{k!} r^{-2}(1 - m^k) \tag{2.10}$$

after a straightforward calculation [3] reproduced in appendix A, one obtains $F(\theta)$ as the sum of a regular and a singular part:

$$F(\theta) = F_{\text{reg}}(\theta) + F_{\text{sing}}(\theta)$$

$$F_{\text{reg}}(\theta) = \sum_{k=1}^{\infty} \frac{\varphi_k \theta^k}{1 - r^{-2}m^k} \tag{2.11}$$

$$F_{\text{sing}}(\theta) = \sum_{j=-\infty}^{+\infty} r^{-2j} \varphi_{\text{rem}}(m^j \theta) = r^{-2}F_{\text{sing}}(m\theta)$$

where

$$\varphi_{\text{rem}}(\theta) = \sum_{k=n}^{\infty} \varphi_k \theta^k \qquad n \geqslant \frac{\ln r^2}{\ln m}. \tag{2.12}$$

The singular part, which has the form given in (1.1) with $a = r^2$ and $\mu = m$, behaves as

$$F_{\text{sing}}(\theta) = A(\theta)\theta^x \qquad x = \frac{\ln r^2}{\ln m} \tag{2.13}$$

where, according to equations (2.4), (2.5) and (2.7), (2.8), the exponent x is related to the surface magnetization exponent β_s through [16]

$$\beta_s = \frac{1-x}{2} = \frac{1}{2} - \frac{\ln r}{\ln m} \tag{2.14}$$

when $F_{\text{sing}}(\theta)$ provides the leading contribution to $F(\theta)$, i.e. when $x < 1$ (see below).

The amplitude $A(\theta)$, which is log-periodic in θ with period $\ln m$, may be written as the Fourier expansion

$$A(\theta) = \sum_{s=-\infty}^{+\infty} A_s \exp\left(2\pi i s \frac{\ln \theta}{\ln m}\right) = \theta^{-x} \sum_{j=-\infty}^{+\infty} r^{-2j} \varphi_{\text{rem}}(m^j \theta). \tag{2.15}$$

The last relation can be inverted and after some algebra, the Fourier coefficients are obtained as [3]

$$A_s = \frac{1}{\ln m} \int_0^{+\infty} du\, u^{-1-x-2\pi i s/\ln m} \varphi_{\text{rem}}(u). \tag{2.16}$$

The system displays two different regimes depending on the value of the modulation amplitude r.

When $r > r_c = \sqrt{m}$, from (2.13) one obtains $x > 1$ and the regular part in (2.11) gives a leading contribution to $F(\theta)$ which is linear in θ. This linear term corresponds to a θ-independent leading contribution to $T(\lambda, r)$ according to (2.8) and

$$S(\lambda, r) = 2 + \frac{m-1}{r^2 - m} + A(\theta)\theta^{x-1} + O(\theta). \tag{2.17}$$

It follows that there is surface order at the critical point with $m_{\text{sc}} \sim (r - r_c)^{1/2}$ and a first-order surface transition [16]. The deviation from the critical magnetization behaves as θ^{x-1} with an oscillating amplitude for $\sqrt{m} < r < m$ and is linear in θ above.

When r takes the form $m^{n/2}$ so that x is equal to the integer n, a logarithmic behaviour is obtained. This is the case in particular at r_c when the surface transition changes from first to second order. Let us write $x = n - \epsilon$ and extract from $F_{\text{reg}}(\theta)$ the term $k = n$ which is now singular, so that

$$F(\theta) = F'_{\text{reg}}(\theta) - \frac{\varphi_n \theta^n}{\epsilon \ln m} + A(\theta)\theta^n(1 - \epsilon \ln \theta). \tag{2.18}$$

The leading contribution to the amplitude $A(\theta)$ comes from A_0, the first term in $\varphi_{\text{rem}}(u)$ leading to a singular contribution at the lower limit in (2.16). Introducing a cut-off at θ_0, one obtains

$$A_0 \simeq \frac{\varphi_n}{\ln m} \int_0^{\theta_0} du\, u^{-1+\epsilon} = \frac{\varphi_n}{\ln m} \frac{\theta_0^\epsilon}{\epsilon} \tag{2.19}$$

which finally gives

$$F(\theta) = F'_{\text{reg}}(\theta) - \varphi_n \theta^n \frac{\ln \theta}{\ln m} + O(\theta^n). \tag{2.20}$$

As a consequence, at r_c the magnetization vanishes as

$$m_s = \left(\frac{m \ln m}{m - 1}\right)^{1/2} (-\ln \theta)^{-1/2} + O(1). \tag{2.21}$$

When $1 < r < r_c$, which corresponds to $0 < x < 1$, the remainder starts on $n = 1$ and since φ_0 in (2.10) vanishes, we have $\varphi_{\text{rem}}(\theta) \equiv \varphi(\theta)$. Inserting $\varphi(\theta)$ given by (2.8) in the amplitude A_s in (2.16), after an integration by parts, one obtains

$$\begin{aligned}
A_s &= \frac{r^{-2}(e^{-mu} - e^{-u})u^{-x-2\pi i s/\ln m}}{\ln r^2 + 2\pi i s}\bigg|_0^{+\infty} + \int_0^{+\infty} du\, \frac{r^{-2}(me^{-mu} - e^{-u})u^{-x-2\pi i s/\ln m}}{\ln r^2 + 2\pi i s} \\
&= \frac{1 - r^{-2}}{\ln r^2 + 2\pi i s} \Gamma\left(1 + \frac{\ln r^{-2} - 2\pi i s}{\ln m}\right).
\end{aligned} \tag{2.22}$$
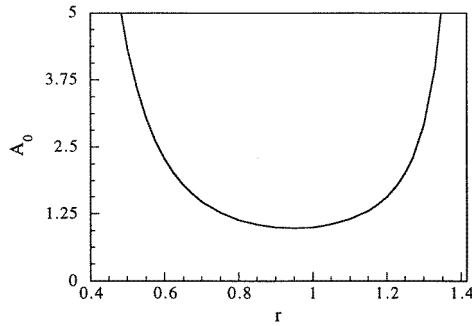
Figure 1. Variation of the leading amplitude A_0 with the modulation amplitude r for the Fredholm sequence with $m = 2$. It diverges at $r_c = \sqrt{2}$ when the surface transition changes from second to first order (see equation (2.19)).
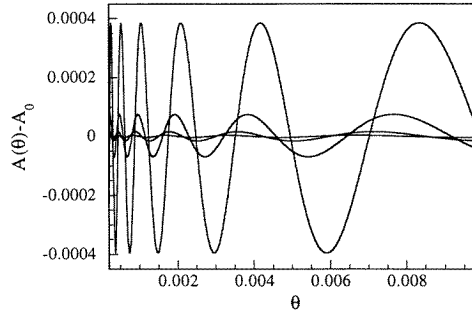
Figure 2. Log-periodic amplitude $A(\theta)$ as a function of the deviation from the critical coupling, $\theta = \ln \lambda^2$, for the Fredholm sequence with $m = 2$. The oscillation amplitude is quickly decreasing when r increases ($r = 0.5, 0.6, 0.7, 0.8$).

The oscillating amplitudes involve the Euler gamma function of a complex argument $\Gamma(z)$. In this regime, as well as for smaller values of r, the behaviour of the surface magnetization is governed by $F_{\text{sing}}(\theta)$. It gives a contribution θ^{x-1} to S which is divergent at the critical point. The surface transition is second order and the magnetization exponent is given by (2.14).

Finally when $r < 1$, $x < 0$, $n = 0$ and once more $\varphi_{\text{rem}}(\theta) \equiv \varphi(\theta)$. The leading power θ^x has an oscillating amplitude and one obtains

$$
\begin{aligned}
A_s &= \frac{r^{-2}}{\ln m} \int_0^{+\infty} du\, (e^{-u} - e^{-mu}) u^{-1-x-2\pi i s / \ln m} \\
&= \frac{r^{-2} - 1}{\ln m} \Gamma\left(\frac{\ln r^{-2} - 2\pi i s}{\ln m}\right).
\end{aligned}
\tag{2.23}
$$

Explicit expressions for the oscillating amplitudes are given in appendix B. The behaviour of A_0 as a function of r for the Fredholm sequence with $m = 2$ is shown in figure 1 whereas the log-periodic oscillations are shown in figure 2 for different values of the modulation amplitude.

3. Critical surface magnetization with a relevant aperiodic sequence

We now consider a sequence generated through substitution on the digits 1 and 0 with
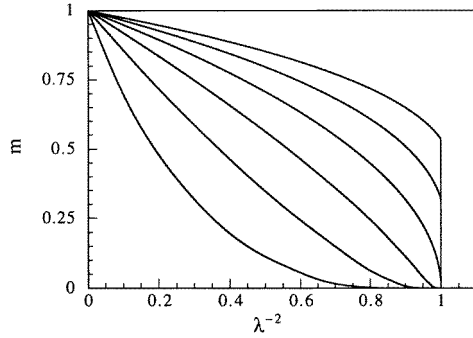
$$
\begin{aligned}
1 &\to \mathcal{S}(1) = \overbrace{1\,1\dots1}^{m}\,0\dots0 \\
0 &\to \mathcal{S}(0) = \underbrace{0\,0\,0\dots0\,0\,0}_{p}.
\end{aligned}
\tag{3.1}
$$

Starting on 1 with $p = 3$ and $m = 2$, the substitution process generates the following sequence:

$$
1\,1\,0\,1\,1\,0\,0\,0\,0\,1\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\dots.
\tag{3.2}
$$

The f_ks satisfy $f_{pk+l} = f_{k+1}$ for $l = 1, m$ and vanish for $l = m + 1, p$ which leads to

$$
n_{pk+l} = \begin{cases} mn_k + lf_{k+1} & \text{for} \quad l = 1, m - 1 \\ mn_{k+1} & \text{for} \quad l = m, p. \end{cases}
\tag{3.3}
$$

Figure 3. Surface magnetization m_s as a function of λ^{-2} for the relevant aperiodic sequence with $p = 4$, $m = 2$ and $r = 1.4$, 1.2, 1, 0.8, 0.6, 0.4 from top to bottom. The surface transition is first-order when $r > 1$ and continuous with an essential singularity when $r < 1$.

The last relation with $l = p$ can be iterated to give $n_L = m^j$ on a sequence with length $L = p^j$. The density of defects $n_L/L = (m/p)^j$ vanishes asymptotically as $L^{\omega-1}$ where

$$\omega = \frac{\ln m}{\ln p} \tag{3.4}$$

is the wandering exponent of the sequence. It follows that the aperiodic modulation does not change the critical point of the Ising quantum chain and its bulk critical behaviour. For $m > 1$, the wandering exponent is positive and, according to Luck's criterion, the sequence gives a relevant perturbation. As a consequence one may expect a change in the surface critical behaviour.

Some general results are known about the critical behaviour of the surface magnetization in the case of a relevant aperiodic perturbation [23]. When the couplings are weakened near to the surface (here for $r < 1$), as shown in figure 3, the surface magnetization vanishes with an essential singularity: $m_s \sim \exp[-ct^{-\omega/(1-\omega)}]$ where t is the deviation from the critical coupling and ω the wandering exponent given in (3.4). When they are strengthened ($r > 1$) there is surface order at the bulk critical point and the critical magnetization vanishes at $r_c = 1$ like

$$m_{sc} \sim (r - 1)^{1/2\omega} . \tag{3.5}$$

Let us now calculate the oscillating amplitude of the critical surface magnetization† by considering the recursion relation for $T(r)$ which is defined as

$$T(r) = S(\lambda_c, r) - 1 = \sum_{j=1}^{\infty} r^{-2n_j} . \tag{3.6}$$

Writing $n_j = n_{pk+l}$, the sum over j is replaced by a double sum: one over $k = 0, \infty$ and the other over $l = 1, p$. Making use of (3.3) one obtains the one-parameter recursion

$$T(r) = \frac{1}{r^2 - 1} - \frac{m}{r^{2m} - 1} + pT(r^m) . \tag{3.7}$$

With $\theta = \ln r^2 \sim r - r_c$, we recover the linear renormalization (2.6). The recursion relation (3.7) may be rewritten as (2.9) with r^{-2} replaced by p and

$$F(\theta) = T(e^{\theta/2}) \qquad \varphi(\theta) = \frac{1}{e^\theta - 1} - \frac{m}{e^{m\theta} - 1} = \sum_{k=0}^{\infty} \varphi_k \theta^k . \tag{3.8}$$

† Outside the critical point, a two-parameter recursion relation would be obtained.
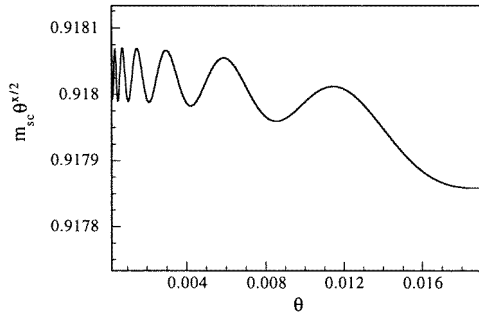
Figure 4. Amplitude of the critical magnetization m_{sc} as a function of the deviation $\theta = \ln r^2$ from $r_c = 1$ for the relevant aperiodic sequence with $p = 4$ and $m = 2$. Near r_c one obtains log-periodic oscillations around $A_0^{-1/2} = \sqrt{12 \ln 2}/\pi$. The influence of the correction terms in (3.12) are visible far from r_c.

The remainder $\varphi_{rem}(\theta)$, which contains the terms in $\varphi(\theta)$ such that $pm^k \geqslant 1$, is $\varphi(\theta)$ itself so that we have

$$F_{reg}(\theta) = \sum_{k=0}^{\infty} \frac{\varphi_k \theta^k}{1 - pm^k}$$

$$F_{sing}(\theta) = \sum_{j=-\infty}^{+\infty} p^j \varphi(m^j \theta) = p F_{sing}(m\theta). \tag{3.9}$$

The singular part diverges as $A(\theta)\theta^x$ with $x = -\ln p / \ln m$ and the critical magnetization behaves as

$$m_{sc} \sim (r - r_c)^{\frac{1}{2} \ln p / \ln m} \tag{3.10}$$

in agreement with (3.4) and (3.5).

Finally, with $\varphi_{rem}(\theta) \equiv \varphi(\theta)$ given by (3.8), the Fourier coefficients in (2.16) are now given by

$$
\begin{aligned}
A_s &= \frac{1}{\ln m} \int_0^{+\infty} du\, u^{-1+\frac{\ln p - 2\pi i s}{\ln m}} \left(\frac{1}{e^u - 1} - \frac{m}{e^{mu} - 1} \right) \\
&= \frac{1}{\ln m} \sum_{k=1}^{\infty} \int_0^{+\infty} du\, u^{-1+\frac{\ln p - 2\pi i s}{\ln m}} (e^{-ku} - me^{-kmu}) \\
&= \frac{p - m}{p \ln m} \Gamma \left(\frac{\ln p - 2\pi i s}{\ln m} \right) \zeta \left(\frac{\ln p - 2\pi i s}{\ln m} \right)
\end{aligned}
\tag{3.11}
$$

where $\zeta(z)$ is the Riemann zeta function. The complete expression of the log-periodic amplitude is given in appendix B.

Collecting these results, one obtains

$$S(\lambda_c, r) = A(\theta)\theta^{-\ln p / \ln m} + \frac{2p - m - 1}{2(p - 1)} + O(\theta). \tag{3.12}$$

The amplitude of the surface magnetization showing the influence of the correction terms is given in figure 4.

4. Conclusion

Some exact results have been obtained for the oscillating amplitude of the surface magnetization of two quantum Ising chains with an aperiodic modulation of the couplings. The two sequences lead to surface-extended perturbations which do not change the bulk critical behaviour.

For the marginal generalized Fredholm sequence, there is surface order at the critical point for $r > r_c = \sqrt{m}$. The critical magnetization vanishes as $(r - r_c)^{1/2}$. The deviation from the critical magnetization, behaving as $t^{\beta'_s}$ with $\beta'_s = 2(\ln r / \ln m) - 1$, has an oscillating amplitude. At r_c, the magnetization vanishes as $(-\ln t)^{-1/2}$. When $r < r_c$, the transition is second order with a continuously varying exponent $\beta_s = \frac{1}{2} - (\ln r / \ln m)$ and a log-periodic amplitude involving the gamma function of a complex argument.

With the relevant sequence, the surface is ordered at the critical point for $r > r_c = 1$. The critical magnetization vanishes as $(r - r_c)^{1/(2\omega)}$ where $\omega = \ln m / \ln p$ is the wandering exponent of the aperiodic sequence. The amplitude of the critical magnetization is log-periodic. It contains a product of gamma and zeta functions of the same complex argument. For $r < 1$, the surface magnetization vanishes with an essential singularity as a function of t.

These aperiodic quantum chains may be considered as discrete realizations of the Hilhorst–van Leeuwen model [24] for which the couplings, $\lambda_k = \lambda(1 + g/k^y)$, decay continuously as a power of the distance from the surface. The decay exponent y corresponds to $1 - \omega$, where ω is the wandering exponent of the sequence.

For both sequences the leading surface critical behaviour is the same as for the Hilhorst–van Leeuwen model [22, 24] (with the correspondence $g \to \ln r / \ln m$ for the marginal sequence [16]). The difference lies in the occurrence of log-periodic critical amplitudes for the aperiodic systems, which is linked to the explicit breaking of the continuous dilatation symmetry introduced by the modulations.

Finally, one may notice that for the Fredholm sequence, the critical magnetization does not show any oscillating amplitude. The same is true for the Rudin–Shapiro sequence which leads to a relevant perturbation [23]. Although discrete scale invariance is necessary for the occurrence of log-periodic critical amplitudes, it is not sufficient.

Acknowledgment

Appendix A. Splitting $F(\theta)$ into regular and singular parts

Inserting the series expansion (2.10) for $\varphi(\theta)$ into (2.9) gives

$$F(\theta) = \sum_{j=0}^{\infty} r^{-2j} \sum_{k=1}^{\infty} \varphi_k m^{jk} \theta^k . \tag{A.1}$$

The order of the two sums can be changed when the sum over j, which is a geometric series, converges. This occurs for $r^{-2} m^k < 1$, i.e. for $k < n$ defined in (2.12). Rewriting $\varphi(\theta)$ as

$$\varphi(\theta) = \sum_{k=1}^{n-1} \varphi_k \theta^k + \varphi_{\text{rem}}(\theta) \tag{A.2}$$

one may split $F(\theta)$ into two parts such that

$$F(\theta) = \sum_{k=1}^{n-1} \frac{\varphi_k \theta^k}{1 - r^{-2} m^k} + \sum_{j=0}^{\infty} r^{-2j} \varphi_{\text{rem}}(m^j \theta) \tag{A.3}$$

where the remainder, $\varphi_{\text{rem}}(\theta)$ given in (2.12), contains that part of the expansion leading to a divergent geometric series in (A.1). The last sum can be extended from $-\infty$ to $+\infty$ by adding and subtracting

$$\sum_{j=-\infty}^{-1} r^{-2j}\varphi_{\text{rem}}(m^j\theta) = \sum_{j=1}^{+\infty} r^{2j}\sum_{k=n}^{\infty}\varphi_k m^{-jk}\theta^k = -\sum_{k=n}^{\infty}\frac{\varphi_k\theta^k}{1-r^{-2}m^k}. \tag{A.4}$$

In this way one obtains

$$F(\theta) = \sum_{k=1}^{\infty}\frac{\varphi_k\theta^k}{1-r^{-2}m^k} + \sum_{j=-\infty}^{+\infty} r^{-2j}\varphi_{\text{rem}}(m^j\theta) \tag{A.5}$$

leading to (2.11).

Appendix B. Explicit results for the oscillating amplitudes

Let us write the Euler gamma function of a complex argument as $\rho_s e^{i\alpha_s}$.

With the Fredholm sequence, when $1 < r < r_c$ equations (2.15) and (2.22) lead to

$$A(\theta) = A_0 + \sum_{s=1}^{\infty}\frac{(1-r^{-2})\rho_s}{\sqrt{\ln^2 r + \pi^2 s^2}}\cos\left(2\pi s\frac{\ln\theta}{\ln m} + \alpha_s - \arctan\frac{\pi s}{\ln r}\right)$$
$$A_0 = \frac{1-r^{-2}}{2\ln r}\Gamma\left(1 - \frac{\ln r^2}{\ln m}\right) \tag{B.1}$$

with [25]:

$$\rho_s = \Gamma\left(1 - \frac{\ln r^2}{\ln m}\right)\prod_{k=0}^{\infty}\left[1 + \left(\frac{2\pi s}{(k+1)\ln m - \ln r^2}\right)^2\right]^{-1/2}$$
$$\alpha_s = \frac{2\pi s}{\ln m}\psi\left(1 - \frac{\ln r^2}{\ln m}\right) + \sum_{k=0}^{\infty}\left[\frac{2\pi s}{(k+1)\ln m - \ln r^2} - \arctan\left(\frac{2\pi s}{(k+1)\ln m - \ln r^2}\right)\right] \tag{B.2}$$

where ψ is the digamma function.

When $r < 1$ equations (2.15) and (2.23) give

$$A(\theta) = A_0 + 2\frac{r^{-2}-1}{\ln m}\sum_{s=1}^{\infty}\rho_s\cos\left(2\pi s\frac{\ln\theta}{\ln m} + \alpha_s\right)$$
$$A_0 = \frac{r^{-2}-1}{\ln m}\Gamma\left(\frac{\ln r^{-2}}{\ln m}\right) \tag{B.3}$$

where now

$$\rho_s = \Gamma\left(\frac{\ln r^{-2}}{\ln m}\right)\prod_{k=0}^{\infty}\left[1 + \left(\frac{2\pi s}{k\ln m + \ln r^{-2}}\right)^2\right]^{-1/2}$$
$$\alpha_s = \frac{2\pi s}{\ln m}\psi\left(\frac{\ln r^{-2}}{\ln m}\right) + \sum_{k=0}^{\infty}\left[\frac{2\pi s}{k\ln m + \ln r^{-2}} - \arctan\left(\frac{2\pi s}{k\ln m + \ln r^{-2}}\right)\right]. \tag{B.4}$$

With the relevant sequence, the Fourier expansion in (2.15) together with (3.11) gives

$$A(\theta) = A_0 + 2\frac{p-m}{p\ln m}\sum_{k=1}^{\infty}\eta_k\sum_{s=1}^{\infty}\rho_s\cos\left(2\pi\frac{\ln\theta}{\ln m} + \alpha_s + \chi_{sk}\right)$$
$$A_0 = \frac{p-m}{p\ln m}\Gamma\left(\frac{\ln p}{\ln m}\right)\zeta\left(\frac{\ln p}{\ln m}\right) \tag{B.5}$$

where

$$\eta_k = k^{-\ln p / \ln m} \qquad \chi_{sk} = 2\pi s \frac{\ln k}{\ln m} \tag{B.6}$$

whereas ρ_s and α_s are now given by (B.4) with r^{-2} replaced by p.

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